Existence results for super-Liouville equations

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The problem

The **Liouville equation** on a closed surface M

$$-\Delta u = \widetilde{K} e^{2u} - K$$

arises in geometry and physics:

- prescribed Gaussian curvature problem
  [Kazdan - Warner, Chang - Yang, Troyanov]
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The supersymmetric extension: **super-Liouville equations**

$$\begin{cases} -\Delta u = \widetilde{K} e^{2u} - K + \rho\, e^{u} |\psi|^2 \\ \not{D}\psi = \rho\, e^{u} \psi \end{cases}$$

$\not{D}$ is the Dirac operator acting on spinors ψ, ρ is a coupling constant.

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Look for $\not{D}$ such that

$$\not{D}^2 = -\Delta_{\mathbb{R}^2} = -\sum_{i=1}^{2} \left(\frac{\partial}{\partial x_i} \right)^2 .$$

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$$\not{D}^2 = \sum_{i=1}^{2} \gamma_i^2 \left(\frac{\partial}{\partial x_i} \right)^2 + (\gamma_1 \gamma_2 + \gamma_2 \gamma_1) \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} .$$

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Take

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

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Hence a spinor is $\psi = \begin{pmatrix} f \\ g \end{pmatrix} : \mathbb{R}^2 \to \mathbb{C}^2$ and

$$\not{D}\psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} f_{x_1} \\ g_{x_1} \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} f_{x_2} \\ g_{x_2} \end{pmatrix} = 2 \begin{pmatrix} g_{\bar{z}} \\ f_z \end{pmatrix}.$$

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It holds $\mathrm{Spec}(\not{D}) = \{\lambda_i\}$ with

$$-\infty \leftarrow \cdots \leq \lambda_{-k} \leq \cdots \leq \lambda_{-1} \leq 0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty$$

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and hence the **associated functional**

$$J_\rho(u, \psi) = \int_M \left(|\nabla u|^2 + 2Ku - \widetilde{K} e^{2u} + \underbrace{2 \left\langle (\not{D} - \rho\, e^u)\psi, \psi \right\rangle}_{strongly\ indefinite} \right).$$

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Known results

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First existence results.

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$$\int_M \widetilde{K} e^{2u} = \int_M K = 2\pi\chi(M).$$

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$$\int_M \widetilde{K} e^{2u} = \int_M K = 2\pi\chi(M).$$

- $\chi(M) < 0 : \quad \widetilde{K} = -1$ *(coercive case)*
- $\chi(M) = 0 : \quad \widetilde{K} = 0$
- $\chi(M) > 0 : \quad \widetilde{K} = 1$ *(non-coercive case)*

Existence results

Let $\chi(M) < 0$. Then

$$\overbrace{-\Delta u = -e^{2u} + 1}^{coercive} + \rho\, e^{u} |\psi|^2$$
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Trivial solution: $(u, \psi) = (0, 0)$.

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Theorem [J.-Malchiodi-Wu, **TAMS (2020)**]

Suppose $\ker(\not{D}) = \emptyset$ and $\rho \notin \mathrm{Spec}(\not{D})$. Then, there exists a **non-trivial solution**.

Nehari-type manifold

Recall $\mathrm{Spec}(\not{D}) = \{\lambda_i\}$ is such that

$$-\infty \leftarrow \cdots \leq \lambda_{-k} \leq \cdots \leq \lambda_{-1} \leq 0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty$$

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Let $\{\varphi_j = \text{negative eigenspinors}\}$. To kill the negative directions we consider the **constraint**

$$N = \left\{ (u, \psi) \ : \ \int_M \left\langle D\!\!\!/\psi - \rho\, e^u \psi, \varphi_j \right\rangle = 0 \quad \forall j < 0 \right\}.$$

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For a constr. crit. pt. (u_0, ψ_0) and a Lagrange mult. μ, we have

$$\begin{aligned} 0 &\overset{constr.}{=} \int_M \left\langle (\not{D} - \rho\, e^{u_0})\psi_0, \varphi_j \right\rangle \\ &\overset{crit.pt.}{=} \int_M \left\langle (\not{D} - \rho\, e^{u_0}) \boxed{\mu\, \varphi_j}, \varphi_j \right\rangle \\ &= \mu \underbrace{\left(-C\|\varphi_j\|^2 - \int_M \rho\, e^{u_0} |\varphi_j|^2 \right)}_{<0}. \end{aligned}$$

Thus $\mu = 0$.

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$$\int_M \langle(\not{D} - \rho)\psi, \psi\rangle \overset{constr.}{\simeq} -\sum_{0<\lambda_j<\rho} (\rho - \lambda_j)\psi_j^2 + \sum_{\lambda_j>\rho} (\lambda_j - \rho)\psi_j^2.$$

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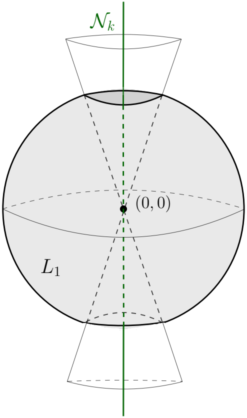

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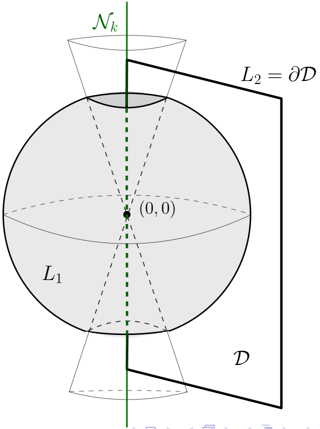

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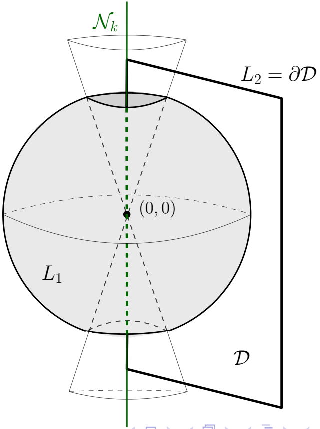

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Any $\rho \in \mathrm{Spec}(\not{D})$ is a **bifurcation point**.

Strategy: use a flow, for ρ passing through $\mathrm{Spec}(\not{D})$, to study the change of topology of the sublevels of the functional J_ρ around $(0, 0)$.

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We consider here a Nehary-type manifold

$$N = \left\{ \text{kill negative directions}, \ \text{center of mass}(u) := \int_{\mathbb{S}^2} \vec{x}\, e^{2u} = 0 \right\}.$$

Other models

Super-sinh-Gordon equations

$$\begin{cases} -\Delta u = -\sinh(2u) + \rho \sinh(u)|\psi|^2 \\ \not{D}\psi = \rho \cosh(u)\psi. \end{cases}$$

Theorem [J.-Malchiodi-Wu, **JDE (2021)**]

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Super-Toda system: w.i.p.

Thank you for your attention!